5

GEOMETRICAL CONSIDERATIONS

The purpose of this chapter is to consider in more detail some of the geometrical properties of space-times that contain plane waves. Part of the discussion here follows naturally from that at the end of the previous chapter. Here, some more general results of a geometrical nature will be considered, and an intuitive approach to the collision of plane waves will also be developed.

5.1 The focusing of congruences

In a general space-time, consider a family of null geodesics. The tetrad vector l^μ may be aligned with the tangent vector field of this congruence. It is also possible to choose an affine parameter along the congruence. In this case, the spin coefficients κ and ϵ are zero, and the first two of the Newman–Penrose equations (2.14*a,b*) are

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \tag{5.1a}$$

$$D\sigma = \sigma(\rho + \bar{\rho}) + \Psi_0 \tag{5.1b}$$

where D is the directional derivative along the congruence. Assume initially that the congruence starts in a vacuum region of space-time with Φ_{00} and Ψ_0 both zero, and that the geodesics are parallel having zero contraction, twist and shear, so that ρ and σ are both zero. Equations (5.1) are identically satisfied.

Now assume that this congruence enters a region containing matter, for which Φ_{00} is non-zero. Since the energy is non-negative, $\Phi_{00} \geq 0$, and equation (5.1*a*) implies that ρ must become increasingly positive. This indicates that the congruence must start to contract. Eventually this congruence will focus. These properties have been described in detail by Penrose (1966). A congruence passing through a region of non-zero energy density is focused by it. It is even possible to measure the magnitude of the energy density by its focusing power.

Consider also an alternative situation in which the congruence enters a region with non-zero Ψ_0. Effectively this means that the congruence meets an opposing gravitational wave. In this case equation (5.1*b*) implies that the congruence starts to shear, the shear axis being determined by the polarization of the gravitational wave. This introduces the non-negative

term $\sigma\bar{\sigma}$ into equation (5.1a), and this in turn induces the congruence to contract. This process has been interpreted by Penrose (1966) in terms of the astigmatic focusing of the congruence by the gravitational wave. He has suggested that the amplitude and polarization of a gravitational wave could also be measured by its focusing properties.

These properties can easily be demonstrated using the plane wave solutions described in the previous chapter, though it is convenient here to define the wave fronts by v =constant, rather than u =constant. This enables us to consider a null congruence which extends into the wave, and on which u, x and y are constant.

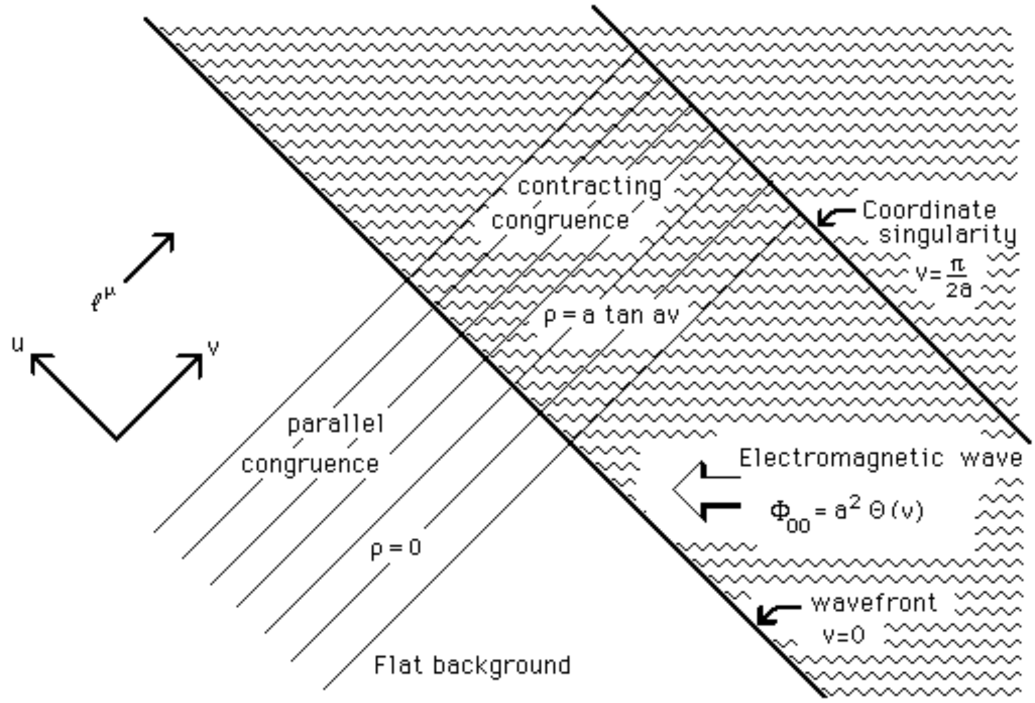

Figure 5.1 An initially parallel congruence meets an opposing electromagnetic wave and is focused by it. The wave front is taken to be the null hypersurface $v = 0$, and the contraction of the opposing congruence becomes unbounded on the hypersurface $v = \pi/2a$. The focusing occurs in the x–y-plane which is not indicated in the picture.

Consider first a parallel congruence in a flat region of space-time which extends into an opposing electromagnetic shock wave as illustrated in Figure 5.1. It is convenient to choose the tangent vector of this congruence to be l^μ. The wave front of the electromagnetic field may be taken to be $v = 0$, so that the energy-momentum tensor of the field is given by $\Phi_{00} = a^2\Theta(v)$. This describes an electromagnetic step wave with a propagation vector n^μ. The complete space-time can be described by the

line element (4.18), but with v replacing u. Transforming this to Rosen form, the flat region in which $v < 0$ has the line element (4.19), and the region $v \geq 0$ which contains the electromagnetic wave is described by

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 av(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{5.2}$$

The tetrad may be taken to be

$$l_\mu = u_{,\mu}, \quad n_\mu = v_{,\mu}, \quad m_\mu = \tfrac{1}{\sqrt{2}}\cos av(x_{,\mu} + iy_{,\mu}) \tag{5.3}$$

and the only non-zero spin coefficient is

$$\rho = a\tan av. \tag{5.4}$$

This indicates that the congruence tangent to l^μ increasingly contracts after passing the electromagnetic wave front. This clearly illustrates a case of pure focusing. Notice that the contraction becomes unbounded on the hypersurface $av = \pi/2$ on which the metric is singular. The focal plane thus coincides with the coordinate singularity of the metric. This is illustrated in Figure 5.1.

Now consider the alternative situation in which the parallel null congruence meets an opposing gravitational shock wave as illustrated in Figure 5.2. Using a similar notation, the gravitational wave can be described by $\Psi_0 = a^2\Theta(v)$, having the wave front $v = 0$. The complete space-time is described by the line element (4.22), but again with v replacing u. In Rosen form, the flat region $v < 0$ has the line element (4.19), and the region $v \geq 0$ which contains the gravitational wave is now described by

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \cos^2 av\,\mathrm{d}x^2 - \cosh^2 av\,\mathrm{d}y^2. \tag{5.5}$$

Using the tetrad

$$l_\mu = u_{,\mu}, \quad n_\mu = v_{,\mu}, \quad m_\mu = \tfrac{1}{\sqrt{2}}(\cos av x_{,\mu} + i\cosh av y_{,\mu}) \tag{5.6}$$

the non-zero spin coefficients are

$$\begin{aligned}\rho &= \tfrac{1}{2}a(\tan av - \tanh av)\\ \sigma &= \tfrac{1}{2}a(\tan av + \tanh av).\end{aligned} \tag{5.7}$$

In this case the congruence tangent to l^μ, on which u, x and y are constants, both contracts and shears after passing the gravitational wave front. This illustrates a case of astigmatic focusing, the shear axes being aligned here with the x and y axes. Notice that both the contraction and

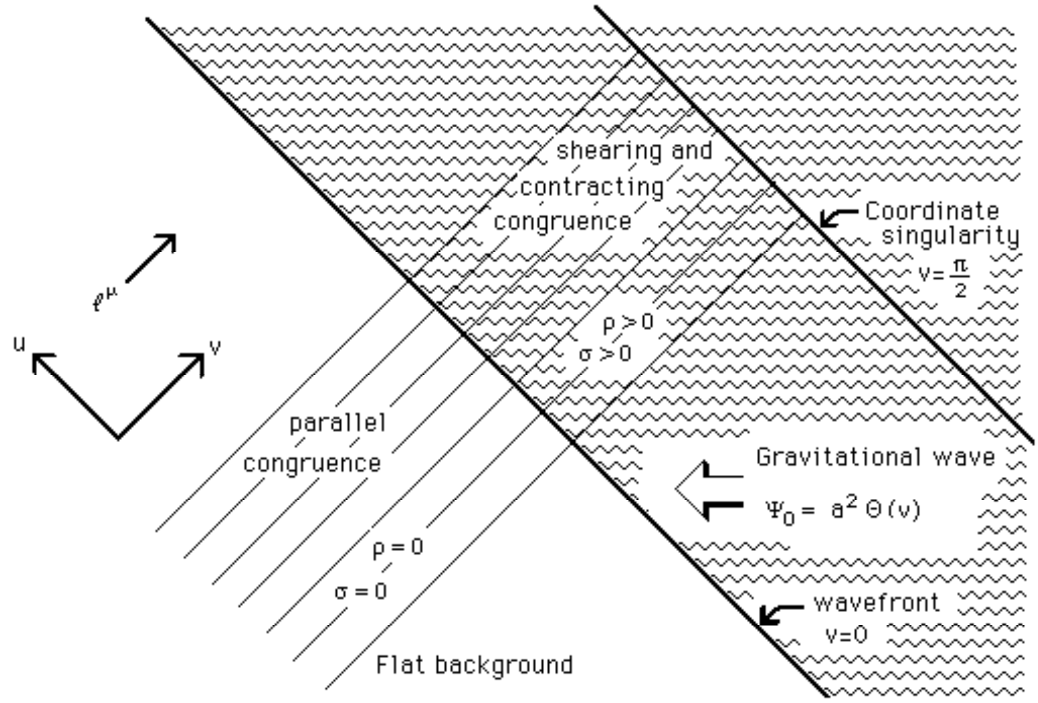

Figure 5.2 An initially parallel congruence meets an opposing gravitational wave and is focused astigmatically. The wave front is given by $v = 0$, and the contraction and shear of the opposing congruence become unbounded as $v \to \pi/2a$. The focusing occurs in the x-direction which, together with the y-direction is not indicated in the picture.

the shear become unbounded on the hypersurface $av = \pi/2$, as indicated in Figure 5.2.

It may also be noticed that, since the polarization vector is here aligned with the x-coordinate direction, this is the direction in which the contraction occurs, producing the singularity in the line element (5.5).

A very similar situation occurs when a parallel congruence meets an impulsive gravitational wave. The fact that the space-time is again flat behind the wave does not alter its focusing properties. The line element equivalent to (4.25) is still singular on the hypersurface on which the opposing congruence is focused.

These general results indicate that parallel congruences that extend into plane waves will tend to produce caustics in the region behind the wave fronts. These caustics, which arise from the focusing properties of the congruences, will normally be associated with coordinate singularities.

5.2 General theorems

There are a number of well known theorems in general relativity that have a bearing on the geometry of congruences and the associated components

of the curvature tensor. It is appropriate to state some of these here as they enable us to build up an intuitive feeling for colliding wave problems.

The best known of these theorems is due to Goldberg and Sachs (1962). It can be stated as follows:

Theorem 5.1 (The Goldberg–Sachs theorem) *A source-free space-time is algebraically special if, and only if, it possesses a shear-free geodesic null congruence.*

It is possible to align the vector field l^μ either with the repeated null direction of the gravitational field, or with the shear-free geodesic null congruence. The Goldberg–Sachs theorem then essentially states that, if $\Phi_{AB}=0$, $\Lambda=0$ then

$$\Psi_0=\Psi_1=0 \quad \Leftrightarrow \quad \sigma=\kappa=0. \tag{5.8}$$

This can easily be proved using Newman–Penrose techniques. Generalizations of this theorem with fewer restrictions on the Ricci tensor have been obtained by Kundt and Trümper (1962), and Kundt and Thompson (1962).

The Goldberg–Sachs theorem can be interpreted as stating that single algebraically special gravitational waves propagate along shear-free geodesic null congruences. The congruences opposing the wave, however, necessarily shear. It has been shown in the above section that, if $\Psi_0\neq 0$, then $\sigma\neq 0$. It follows that the different gravitational wave components cannot be simply superposed.

A related theorem applying to Einstein–Maxwell fields has been obtained by Mariot (1954) and Robinson (1961). This can be stated in the following way.

Theorem 5.2 (The Mariot–Robinson theorem) *The repeated principal null direction of a null electromagnetic field is necessarily tangent to a shear-free geodesic null congruence, and is also a repeated principal null direction of an algebraically special gravitational field.*

If l^μ is aligned with the repeated principal null direction of the electromagnetic field, this theorem can be sumarized by the statement

$$\Phi_0=\Phi_1=0,\ \Phi_2\neq 0 \quad \Rightarrow \quad \sigma=\kappa=0,\ \Psi_0=\Psi_1=0. \tag{5.9}$$

The first part of this theorem, which states that a null electromagnetic field necessarily propagates along a shear-free geodesic null congruence, follows immediately from Maxwell's equations in the form (2.18). The remainder can easily be obtained using Newman–Penrose techniques.

One further theorem is of relevance to the interacting wave problem. This was obtained by Szekeres (1965).

> **Theorem 5.3** (The Szekeres theorem) *No solutions of Einstein's source-free field equations exist for which* $\Psi_0 \neq 0$, $\Psi_4 \neq 0$, *with* $\Psi_1 = \Psi_2 = \Psi_3 = 0$.

This is easily proved using the Newman–Penrose formalism. It follows immediately from Theorem 5.3 that, in a vacuum space-time, two transverse gravitational waves cannot be simply superposed. For the type of interactions considered in this book, in which the approaching gravitational waves are described by the components Ψ_0 and Ψ_4, it is clear that other components of the gravitational field must be induced as part of the interaction between such waves. A similar theorem proving the non-superposition of two longitudinal gravitational waves has been obtained elsewhere (Griffiths 1975*a*).

5.3 Colliding waves

The theorems in the previous section enable us immediately to gain some insight into some of the general properties of colliding wave problems. The basic situation may be considered as described in Figure 3.1, in terms of a flat background in region I, two approaching waves in regions II and III, and an interaction region IV following the collision.

Consider first the collision of a pair of plane gravitational waves. Initially, prior to the collision, each wave is of type N, and propagates along shear-free geodesic null congruences according to the Goldberg–Sachs theorem 5.1. However, congruences in the opposite direction to each wave necessarily shear. Thus, as the two waves meet and begin to pass through each other, they start to shear and contract. They begin to focus each other astigmatically.

It is possible to align the two vectors l^μ and n^μ with the propagation vectors of the approaching gravitational waves. The waves are thus described initially by the components Ψ_4 and Ψ_0 respectively. It is then clear from the Szekeres theorem 5.3, that the interaction region cannot be characterized solely by these two components, additional terms must appear. We have already seen in the Khan–Penrose solution (3.10), that it is the Ψ_2 term which appears. In fact, as will be shown later, this is the usual case.

It may also be worth pointing out that we would expect the gravitational field in the interaction region to be algebraically general. Clearly the two vectors l^μ and n^μ are not aligned with principal null directions of the Weyl tensor in the interaction region, though we may still think

of them as being aligned with the two initial wave components. There are, however, some exceptions and degenerate cases that are algebraically special will be considered later.

Now consider the collision of a pair of electromagnetic waves. In regions II and III, each will propagate along null geodesics having expansion twist and shear all zero. Null congruences passing through them in opposite directions are focused anastigmatically. Thus, as the two waves meet, we would initially expect them to focus each other without the introduction of shear. This, however, is not the case. In fact, it will be shown later that shear terms necessarily appear, and so the two waves must focus each other astigmatically. The introduction of shear, however, consequently introduces non-zero components of the Weyl tensor. In this way, the collision of electromagnetic waves can be considered to generate gravitational waves.

Finally, consider the collision of a gravitational wave with an electromagnetic wave. Let region II contain the initial gravitational wave with component Ψ_4, and let region III contain the approaching electromagnetic wave with component Φ_0. Congruences on which $u =$ constant are focused in region III, where $\rho \neq 0$, but $\sigma = 0$. We would therefore expect the gravitational wave to be focused anastigmatically as it enters the electromagnetic wave. In contrast, congruences on which $v =$ constant are focused astigmatically in region II. The non-zero contraction and shear are here given by $-\mu$ and $-\lambda$. We would therefore expect the electromagnetic wave to be focused astigmatically as it enters the gravitational wave. However, Maxwell's equations (2.18) would then become inconsistent if Φ_0 were the only non-zero component of the electromagnetic field. It therefore follows that other components of the electromagnetic field must appear in the interaction region.

This result is not unexpected. It follows from the Mariot–Robinson theorem 4.2 that a null electromagnetic wave must propagate along a shear-free geodesic null congruence. As the electromagnetic wave meets the gravitational wave, this congruence must start to shear, and therefore the electromagnetic field cannot remain null. The electromagnetic wave must therefore be partially reflected by the gravitational wave. This feature of the backscattering of the electromagnetic wave has been described by Penrose (1972).

We have now concluded that an electromagnetic wave must be partially reflected by a gravitational wave. But what about the gravitational wave: is that also partially reflected? To answer this question we will need to obtain exact solutions. In fact, the gravitational wave is not necessarily reflected. Exact solutions exist in which the Ψ_0 term is not generated in the interaction region, though the Ψ_2 term always appears. In these cases

the Weyl tensor in the interaction region is thus algebraically special. This simplifies the field equations significantly, and enables us to find a fairly general class of exact solutions. Given an arbitrary plane gravitational wave in region II, and a plane electromagnetic wave in region III, an exact solution in region IV can be obtained explicitly.